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GRAPH-LIKE CONTINUA, AUGMENTING ARCS, AND MENGER'S THEOREM

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We show that an adaptation of the augmenting path method for graphs proves Menger's Theorem for wide classes of topological spaces. For example, it holds for locally compact, locally connected, metric spaces, as already known. The method lends itself particularly well to another class of spaces, namely the locally arcwise connected, hereditarily locally connected, metric spaces. Finally, it applies to every space where every point can be separated from every closed set not containing it by a finite set, in particular to every subspace of the Freudenthal compactification of a locally finite, connected graph. While closed subsets of such a space behave nicely in that they are compact and locally connected (and therefore locally arcwise connected), the general subspaces do not: They may be connected without being arcwise connected. Nevertheless, they satisfy Menger's Theorem.

1. Introduction

Menger's Theorem is a cornerstone of graph theory. The origins of the theorem date back to the work of several topologists around the year 1930, including Menger himself. Their results hold for a certain class of topological spaces which has been successively extended, notably by Whyburn [38] in 1948.

Within the graph theory community, this subject is still very much alive. Even for finite graphs, there are several proofs; see for example

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McCuaig [22] and references therein. A new short proof has been given as recently as 2001 [3]. There are several equivalent formulations; see for example Dirac [14]. Menger's Theorem is also a prototype of min-max results on which combinatorial optimization is based.

There are numerous proofs in graph theory where Menger's Theorem is used. Let us mention one example which may also be of interest for more general topological spaces. Let A be a set of k points in a metric space M. Does M contain a simple closed curve containing A? A necessary condition is that, for any set S of fewer than k points, disjoint from A, $M \setminus S$ has a simple arc between some two points of A. A well-known observation of Dirac says that the following condition is sufficient when M is a graph and the k points in A are vertices: for any set S of fewer than k vertices, $M \setminus S$ is connected. The proof is an elegant application of Menger's Theorem. Thus a similar result holds for all the metric spaces which in this paper are shown to satisfy Menger's Theorem.

For infinite graphs there have been two main sources of inspiration to work on Menger's Theorem: the Erdős–Menger Conjecture, now a deep theorem [1], and the Freudenthal compactification of a locally finite graph. The strongest results available in either field do not imply the ones in the other.

In this paper we show that a variant of the augmenting path method used for finite graphs can be used to simultaneously prove Menger's Theorem for wide classes of topological spaces. To demonstrate the generality, we derive the result of Whyburn that the theorem holds for locally connected, locally compact, metric spaces. We also apply the method to a new class of topological spaces introduced by Vella, which we call *graph-like continua*. We show that they are locally connected, and therefore satisfy Menger's Theorem by the results of [28]. Graph-like continua include the Freudenthal compactifications of locally finite graphs. Closed subspaces of the Freudenthal compactification are compact and inherit therefore the nice properties of the Freudenthal compactification itself. But, the general subspaces do not (as shown in [16]). Nevertheless they satisfy Menger's Theorem as we show.

In a series of papers, in particular [5,6,8,11–13], Diestel *et al.* have investigated several issues, including Menger's Theorem and cycle spaces, in the Freudenthal compactification $|G|$ of a locally finite graph G, and in similar spaces, without using local connectedness or compactness. On the other hand, Vella and Richter [36] have adopted a more topological approach to cycle spaces, focusing on compactness in very general topological spaces. Thus our first results (on local connectedness of graph-like continua) give a bridge between some of the more topological results, including the versions of Menger's Theorem from the 1930s, and some of the more graph-theoretic results by Diestel *et al.* Moreover, the result in Diestel [7, Lemma 8.5.4] saying that closed connected subspaces of $|G|$ are arcwise connected, and a more general result of Diestel and Kühn [13], also follow from our result.

In the second part of the paper, the main results are topological versions of Menger's Theorem further extending the class of spaces for which the theorem is known to hold. Whyburn points out that Menger's Theorem is, in general, false for complete, locally connected metric spaces (see Figure 2), and therefore restricts himself to locally compact, locally connected spaces. We deal with some other spaces, replacing the locally compact requirement with a global connectivity requirement which is well-known in topology and which fits particularly nicely with the augmenting path method. More precisely, we prove Menger's Theorem for locally arcwise connected, *hereditarily* locally connected (HLC) metric spaces, such as the one illustrated in Figure 3b. A precise definition of property HLC is given later.

Property HLC comes up in mainstream general topology [20, 27]; it follows from the well-known Moore–Menger Theorem (see [15]) that complete HLC metric spaces are locally arcwise connected, so the property of being complete (that is, every Cauchy sequence is convergent) also fits naturally in our work. Finally, it turns out that graph-like continua in fact satisfy these assumptions (as we shall prove), so that our proof of Menger's Theorem also applies to these spaces. (The fact that Menger's Theorem holds for graph-like continua follows already from our first result that these spaces are locally connected combined with Whyburn's result for locally connected, locally compact spaces.)

1.1. The graph-theoretic context

Menger's Theorem comes in many flavours. A simple, very general formulation for finite graphs is the following:

Let A, B be disjoint sets of vertices in a finite graph, and let k be a natural number. Suppose that deleting k vertices outside $A \cup B$ always leaves an A–B path. Then there exist $k+1$ independent A–B paths.

One twist to the formulation comes from the meaning of the word *independent*: It could simply mean that the paths are *totally disjoint*, that is, having empty intersection (pairwise), or else *internally disjoint*, that is, pairwise disjoint except possibly for their endvertices. In a non-topological setting, the latter version can easily be derived from the former, but the topology makes it a different problem.

Recent advances in infinite and topological graph theory specifically related to Menger's Theorem have focused on the *totally disjoint* version of the theorem, more precisely the Erdős–Menger Conjecture [1,5,6,8]. Less recently, versions of Menger's Theorem in infinite graphs were obtained by Halin [18], Zelinka [40], and Polat [29,30].

It has long been known, and is also emphasized by recent work of Diestel, Richter and their students [4,11,13,35,36] that the natural setting for problems in infinite graphs can be that of a topological space obtained by adding the *ends* as "points at infinity" to the graph. It then becomes natural to talk of (topological) *arcs* instead of (graph-theoretic) *paths*. One special case of our results is a version of Menger's Theorem for any subspace of the Freudenthal compactification of a locally finite graph.

The example of Figure 1, taken from Diestel and Kühn [13], illustrates how non-locally-finite graphs can be problematic from the point of view of Menger's Theorem. Clearly, no single point separates ω and ω'. On the other hand, every simple arc starting at ω must contain a horizontal arc starting at ω, and therefore any two simple arcs from ω to ω' must have more than ω and ω' in common.

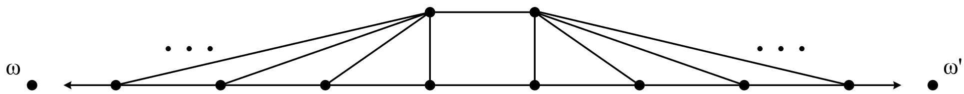


Figure 1. "A Menger problem for ω and ω'."

This problem is essentially the same as the one encountered by topologists, in particular Whyburn, and is the one which defines the boundary of applicability of Menger's Theorem (see example in Figure 2).

1.2. Topological History and background

We recall that a *continuum* is a compact, connected, non-empty Hausdorff space. An excellent reference on continuum theory is Nadler [27]. A finite, non-empty connected graph may be viewed as a metric continuum. Menger's Theorem for these spaces seems to have been obtained first in 1927 by Menger [23]; the original proof was elaborated by Nöbeling, as reported in [2], where Blumenthal and Menger give a proof in English. In [25] Menger gives a historical account of his contribution to the theorem, intended primarily for graph theorists.

A version of Menger's Theorem for locally connected, plane continua was given by Rutt [31], for some other metric continua by Menger [24], and for general locally connected metric continua by Nöbeling [28]. Zippin [41] gave a version for locally compact, locally connected, metric, separable spaces.

Fifteen years later (in 1948) Whyburn gave what appears to have survived as the definitive topological treatment of Menger's Theorem. In a short paper [38], he proved the totally disjoint and the internally disjoint versions of Menger's Theorem under different assumptions. The former he proves for locally connected, separable, complete, metric spaces, but then points out, referring to an example in an earlier paper [37], which we illustrate in Figure 2, that these assumptions do not suffice for the internally disjoint version.[1] (In Figure 2 no single point separates a and b. On the other hand, any two simple arcs from a to b must have more than a and b in common for the same reason as in Figure 1.) He then proves the latter for all locally connected, locally compact metric spaces.

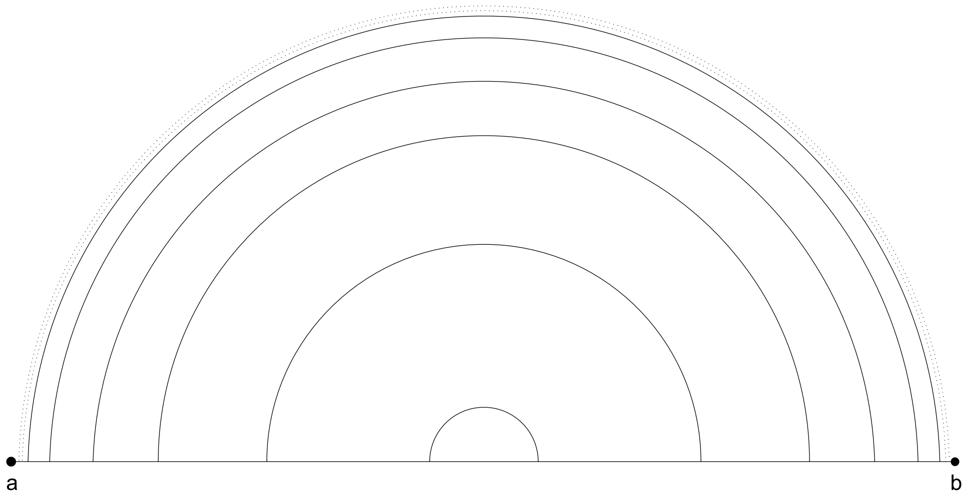


Figure 2. Whyburn's example of a locally connected, completely metrizable space[1] which does not satisfy the conclusion of Menger's Theorem.

The *totally disjoint* version was taken up again in 1971 by Hunt [19] and in 1976 by Tymchatyn [33]; the latter gave a version for T_1 (possibly non-metric) regular, locally connected spaces; this allowed him to use topological quotients to resolve a related issue raised by Hunt. In 1971 Gilbert gave a version for arbitrary Hausdorff spaces, assuming though that the sets A, B are open [17].

[1] Although Whyburn's example inherits its topology from the Euclidean plane, it does not inherit a complete metric. Nevertheless, it is completely metrizable – it suffices to define the distance between two points as the length of the shortest arc (in the space, not the plane) between them.

If we seek a collection of k totally disjoint simple arcs with prescribed ends, we have a topological variant of the *linkage* problem in graph theory. In [32] this problem is solved for the locally 2-connected, compact, metric spaces in the sense that, for each fixed finite k, it is a finite problem to describe the spaces (and the location of the prescribed endpoints) which do not contain the desired arcs.

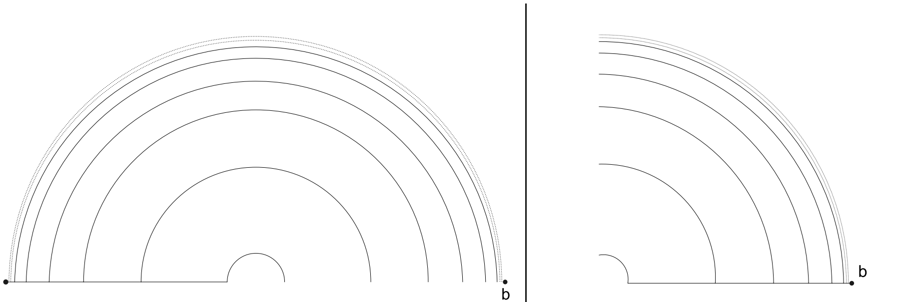
In this paper we unify and extend the results on Menger's Theorem. The method throughout the paper is the augmenting arc method which is a modification of the augmenting path method for graphs. Before we describe the method, we mention some of its limitations. As far as we can see, it does not apply to the linkage problem, and it does not apply to non-metric spaces, and we do not know if such extensions are possible by other methods (except that our proof of Menger's Theorem in the totally disjoint version also works for non-metric spaces). It is also essential that the number of k of arcs be finite, even in the totally disjoint version. To see this, let M denote the square $[0,1]^2$. Let A consist of all points of the form $(0,1/n)$, and let B consist of all points of the form $(1,1-1/n)$, where n is a natural number. Then no finite set separates A and B in M. Yet, M does not contain a collection of infinitely many pairwise disjoint simple arcs from A to B. The subspace consisting of the union of all straight line segments between A and B is another example, and this space is graph-like (to be defined later). Despite this example, Menger's Theorem for infinitely many pairwise disjoint arcs may still hold in specific spaces. Thus we can show by a straightforward (though slightly tedious) argument that the totally disjoint version of Menger's Theorem holds for any two disjoint sets A,B in the Freudenthal compactification of a locally finite graph. However, this argument is purely graph theoretic in that the simple arcs joining A,B can be chosen to be (the closure of) paths in the graph, and thus our argument does not extend to the more general case.

1.3. Terminology and property *HLC*

Definition 1.1. A topological space X is *hereditarily locally connected* if every closed connected subset of X is locally connected. We also say X has property HLC. A topological space is *strongly* HLC if *every* connected subset is locally connected.

Property HLC is a standard property of some importance in continuum theory; see [20,27]. In particular, all dendrites are HLC. It is known that all regular metric continua are HLC [2]; this includes the well-known *Sierpinski* *triangle* [2, p. 476]. It was shown by Wilder [39] that a metric continuum is HLC if and only if it is strongly HLC; in [34] this was extended to arbitrary continua. In 1999 Levin and Tymchatyn [21] answered a question of Duda by showing that every separable, metric, strongly HLC space is sparse; more precisely, it is one-dimensional, that is, every point has arbitrarily small neighbourhoods with a zero-dimensional boundary (see the beginning of Section 2 for a definition of zero-dimensionality).

The graph-like continua we shall introduce in this paper are also HLC. We remark that this is not a local property: Figure 3a illustrates why Whyburn's counterexample in Figure 2 is not HLC. On the other hand, every point in the space of Figure 2 has a neighbourhood which is HLC. For example, the space in Figure 3b shows such a neighbourhood around b. (The referee asked if replacing "connected" by "arcwise connected" in the definition of strongly HLC makes it a local property. The answer is negative. To see this, one just needs to modify the example in Figure 2 by adding another simple arc from a to b. Then the subspaces in Figure 3, modified accordingly, still show that the (modified) example of Figure 2 is not HLC (in the modified sense) but at b is still locally homeomorphic to the (modified) example of Figure 3b, which is still strongly HLC (in the modified sense).)



a) A connected, non-locally-connected subspace of Whyburn's example.

b) An HLC space which is a subspace of Whyburn's example.

Figure 3. Subspaces of Whyburn's counterexample.

The reader will be assumed to be familiar with the very basic terms from topology, such as *metric space*, *diameter* (supremum of distances), *closure*, *interior*, *boundary*, and *open*, *closed*, *compact*, *connected*, *arcwise-connected* sets; see for example [20]. If $\mathcal{P}$ is a topological property, then *locally* $\mathcal{P}$ means that for every point v, and every neighbourhood U around v, there is a neighbourhood of v contained in U with property $\mathcal{P}$. The closure and interior of a set A will be denoted by $\mathtt{cl}(A)$, $\mathtt{int}(A)$, respectively. We shall use the well-known fact that compact, connected, metric spaces are arcwise connected; see, for example, [27, Theorem 8.23].

We also use only very basic terms from graph theory, such as *graph*, *vertex*, *edge*, *path*, *component*, *locally finite* (meaning that all vertices have finite degree or valency), see for example [26]. Henceforth, all other terms needed will be defined, or there will be a specific reference. An *arc* (respectively *semi-arc*) is the continuous image of the unit interval, $[0,1]$ (respectively $[0,1)$), (with the usual topology). A *simple arc* is a homeomorphic image of the closed unit interval. For notational convenience we shall distinguish between the *starting point* or *first point* of the arc and the *ending point* or *terminating point* or *last point*. The arc inherits the order relation of $[0,1]$. This enables us to say that one point of the arc *precedes* or *succeeds* another, and every subset of the arc has a supremum and an infimum. A homeomorphic image of $[0,1)$ is called a *simple semi-arc*. If a,b are distinct points on an arc or semi-arc P, then $P[a,b]$ and $P[a,b)$ denote the arc and semi-arc, respectively, from a to b on P. A *simple closed curve* is a homeomorphic image of a circle.

2. Graph-like continua

In this section we establish basic properties of graph-like continua, which are generalizations of the Freudenthal compactification of a locally finite graph. These results are, hopefully, of independent interest, and they are also needed for Menger's Theorem for these spaces and their subspaces.

We recall that a topological space V is *zero-dimensional* if it has a base of subsets which are both open and closed. Assuming that V is Hausdorff, this implies that for any two points x,y, there exists a separation of V into disjoint open sets each containing one of x,y. Moreover, V is highly fragmented; more precisely, the connected subsets are just the singletons. In fact, for compact Hausdorff spaces, this condition is equivalent to zero-dimensionality. These are the only facts relating to zero-dimensionality that we shall need. All finite metric spaces are zero-dimensional; the Cantor set is perhaps a more interesting example.

Given a topological space G, an *edge* of G is an open subset of G, homeomorphic to the set of real numbers, whose closure is a simple arc. A *graph-like* space is a topological space G equipped with a collection E of pairwise disjoint edges such that $G-E$ is zero-dimensional. We refer to the set $G-E$ as the *vertex set* and denote it by V. Note that the boundary of an edge consists of precisely two points. These two points are vertices, as they cannot be contained in any edge (because an edge is open).

We say that two points in a topological space are *separated* if there exists an open and closed set containing one but not the other. (Recall that a space is connected if no two points are separated.) A *component* is a maximal set of points no two of which are separated (in the relative topology); in other words, a maximal connected subspace. If e is an edge of a connected topological space G, then $G-e$ has at most two components. In fact, either $G-e$ is connected or else it is the disjoint union of two non-empty open and closed sets, C_1 and C_2. Both must be connected, for otherwise $G-e$ is the disjoint union of *three* non-empty sets which are open and closed. One of these is disjoint from $\mathtt{cl}(e)$, and therefore is open and closed in G, contradicting the fact that G is connected. Hence C_1 and C_2 are the components of $G-e$.

Clearly all finite graphs are graph-like; it is easy to see that the Freudenthal compactification of a locally finite graph is graph-like: The edges of the Freudenthal compactification are the edges of the graph, and the vertices of the Freudenthal compactification are the vertices of the graph together with the set of ends of the graph. This also holds for similar constructions obtained from non-locally-finite graphs and studied in recent papers of Diestel *et al.* [9,10,12] (see discussion later in this section).

Whyburn's example shows that graph-like spaces can be badly behaved. Also, such spaces need not be metrizable. A simple example can be constructed by letting the vertices be all ordinals up to and including the first uncountable ordinal, and inserting an edge (a copy of the real line) in between every pair of consecutive ordinals. A basic neighbourhood of a limit ordinal α consists of all ordinals between α and β, where $\beta<\alpha$, including all edges in between these ordinals together with a segment from α to a point on the edge between α and $\alpha+1$. A basic neighbourhood of a non-limit ordinal $\alpha+1$ consists of a segment from $\alpha+1$ to a point on the edge between $\alpha+1$ and $\alpha+2$ together with a segment from $\alpha+1$ to a point on the edge between $\alpha+1$ and α. Note there is no countable neighbourhood base for the uncountable ordinal, so the space is not metric. Under the assumption of compactness and metrizability, however, many difficulties disappear.

Theorem 2.1. *Let G be a graph-like metric continuum. Then G is locally connected.*

Proof of Theorem.

Claim. For every positive real ϵ, there are only finitely many edges with diameter larger than ϵ.

Proof of claim. Suppose there exists an infinite sequence $e_1, e_2, \ldots$ of such edges. Then there exist infinite sequences $p_1, p_2, \ldots, q_1, q_2, \ldots$ such that $p_i, q_i \in$

e_i and $d(p_i,q_i)>\epsilon$. Since G is compact, by taking subsequences if necessary, we may assume that $\{p_i\}_{i\in\mathbb{N}},\{q_i\}_{i\in\mathbb{N}}$ converge to the *distinct* points p,q respectively.

Since edges are open, p and q are vertices. Since V is zero-dimensional, it can be partitioned into disjoint closed sets P,Q containing p,q respectively; since V is closed, P,Q are closed in G, and there exist disjoint open sets P',Q' of G containing P,Q respectively. Then, for sufficiently large i, $p_i\in P'$ and $q_i\in Q'$. However, for all such i, there exists $z_i\in e_i\setminus(P'\cup Q')$, for otherwise $\{P'\cap e_i, Q'\cap e_i\}$ would be a separation of the connected set e_i. By compactness, there exists some subsequence $\{z_{k_i}\}_{i\in\mathbb{N}}$ converging to a vertex z. But $z\in P\cup Q$, so eventually this sequence must fall within P' or Q', a contradiction which proves the claim. ∎

For every positive integer n, we consider the collection $\mathcal{F}_n$ of edges with diameter larger than $1/n$. By the above claim, $\mathcal{F}_n$ is finite. For every $e\in\mathcal{F}_n$ we choose a simple arc $a_e^{(n)}\subseteq e$ such that $e\backslash a_e^{(n)}$ is entirely within distance $1/n$ from the boundary of e. (The existence of $a_e^{(n)}$ follows from the definition of continuity.) We also ensure that $\mathtt{int}(a_e^{(n+1)})\supseteq a_e^{(n)}$. Finally, we denote by X_n,Y_n the subsets $G\setminus\bigcup_{e\in\mathcal{F}_n}\mathtt{int}(a_e^{(n)})$ and $G\setminus\bigcup_{e\in\mathcal{F}_n}a_e^{(n)}$ respectively.

We need to show that for every point $x\in G$ and every ball B with centre x there exists a connected, open neighbourhood of v contained in B. Clearly we may assume that B is closed (and has positive radius, say r). Since edges are locally connected and open, we may assume that v is a vertex.

Since $\mathtt{int}(a_e^{(n)})$ is open and has a boundary consisting of two points, its deletion can increase the number of components by at most one; the same is true of $a_e^{(n)}$, since it is contained in $\mathtt{int}(a_e^{(n+1)})$. Since $\mathcal{F}_n$ is finite, X_n and Y_n have only finitely many connected components. Since Y_n is open, its components are open in G. Let K_n denote the component of X_n containing x. The (open) component of Y_n containing x is contained in K_n. Hence it is sufficient to show that, for some n, K_n is contained in B.

Suppose, by way of contradiction, that for every positive integer n, the set $\{y\in K_n\,|\,d(x,y)>r\}$ is not empty. Then the set $S_n:=\{y\in K_n\,|\,d(x,y)=r\}$ is also not empty, for otherwise $\{y\in K_n\,|\,d(x,y)\le r\}$ and $\{y\in K_n\,|\,d(x,y)\ge r\}$ give a separation of the connected subset K_n.

Note that $V:=\bigcap_{n\in\mathbb{N}}X_n$ is precisely the set of vertices. Moreover, we have $K_1\supseteq K_2\supseteq K_3\cdots$ and $S_1\supseteq S_2\supseteq S_3\cdots$. Since the K_i's are closed and connected and the S_i's are closed and non-empty, we have that $\bigcap_{n\in\mathbb{N}}S_n$ is non-empty, that is, contains a point z, and $K:=\bigcap_{n\in\mathbb{N}}K_n$ is connected.

The point z, being at distance r from x, is distinct from x. Moreover, since for every positive integer n we have $S_n\subseteq K_n$, we have that z,x both belong to K. But $K \subseteq V$, which is zero-dimensional, whence $K = \{x\}$, a contradiction. This completes the proof of the theorem. ∎

Proposition 2.2. *Suppose X is a graph-like metric continuum, and that H a closed connected subset of X. Then H is a graph-like metric continuum.*

Proof. Let V, E be respectively the vertex and edge sets of X. If $H \subseteq \mathtt{cl}(e)$ for some $e \in E$, then H is a singleton or a simple arc, which are graph-like. So we assume otherwise.

For any edge e not contained in H, $e \cap H$ consists of at most two components; for any such component K, $\mathtt{cl}(K)$ is a simple arc because every connected subset of the reals is either a singleton or an interval. The connectedness of H implies that $\mathtt{cl}(K)$ cannot be a singleton. Hence $\mathtt{int}(K)$ satisfies the requirements of an edge, and the boundary of K consists of one point in V, and one point $w(K)$ not in V. We take A to be the collection of edges $\mathtt{int}(K)$, and W the set of points $w(K)$, over all such components K.

Clearly, the only connected subsets of $V \cup W$ are the singletons. Since $V \cup W$, being a closed subset of a compact Hausdorff space, is compact and Hausdorff, $V \cup W$ is zero-dimensional. Hence H is a compact graph-like space with vertex set $(V \cap H) \cup W$ and edge set consisting of A and those edges in X which are contained in H. ∎

Corollary 2.3. *Suppose X is a graph-like metric continuum. Then X is HLC.* ∎

Corollary 2.3 combined with the result of Wilder [39] shows that a graph-like metric continuum is even strongly HLC. Since locally connected, metric continua are arcwise connected, see [27, Theorem 8.23], Theorem 2.1 and Proposition 2.2 also imply the following corollary.

Corollary 2.4. *Suppose X is a graph-like metric continuum. Then every closed, connected subspace of X is arcwise connected.* ∎

For the special case when X is the Freudenthal compactification of a locally finite graph, the above result appears, without proof, as Lemma 8.5.4 in [7]. The proof can be found in Diestel and Kühn [13]. In [13] the authors work at a greater level of generality: They start with a possibly non-locally-finite graph G and, assuming G is countable and *finitely connected* (that is, no two vertices are joined by infinitely many internally disjoint paths), they prove the statement of Corollary 2.4, where the graph-like continuum X is replaced by a special construction which they refer to as $\tilde{G}$, and which coincides with the Freudenthal compactification when G is locally finite.

Although the space $\tilde{G}$ is not a standard construction like the Freudenthal compactification, the question arises whether or not, at this level of generality, the construction $\tilde{G}$ is graph-like. This is not always the case. In fact, Diestel and Kühn themselves [13, Proposition 3.4] give an example of a graph such that the space $\tilde{G}$ contains an entire simple closed curve disjoint from the edges; thus the complement of the edges is certainly not zero-dimensional.

Since these objects are problematic for other reasons, in the same paper Diestel and Kühn go on to impose the slightly stronger restriction that G be *finitely edge-connected*, that is, no two vertices are joined by infinitely many edge-disjoint paths. It is easy to see (as in [36, Lemma 36]) that under this assumption the space $\tilde{G}$ is graph-like. Moreover, it has been shown by Diestel [9], and independently by Vella and Richter [36], that if G is finitely connected and 2-connected (that is G is connected and has no separating vertex), and when the neighbourhoods of the ends of G are defined appropriately, then $\tilde{G}$ is compact. It has also been shown by Vella and Richter [36, Theorem 37] that under the same assumptions, $\tilde{G}$ is metrizable.

Thus, when G is a 2-connected, finitely edge-connected graph, $\tilde{G}$ is a graph-like continuum. Consequently, the fact that closed connected subspaces of $\tilde{G}$ are arcwise connected follows from Corollary 2.4.

We close this section on graph-like continua with a result that we shall need later on. We remark that graph-like continua fall within the more general class of *edge spaces* studied by Vella and Richter [36]. The main ideas in the following proof are to be found in their treatment of *edgecuts*; here we formulate an assertion tailored to our needs, and provide a short self-contained proof.

We say that a metric space M is *finitely separated*[2] if, for any two points x, y in M, there exists a finite separator S separating x and y, that is, some closed and open subset of $M \setminus S$ contains x but not y. We claim that, if M is compact, this implies (and hence is equivalent to) the statement that every point has arbitrarily small neighbourhoods with finite boundary, or equivalently if every closed space can be separated from any (fixed) point it does not contain by a finite set. To prove this claim, choose arbitrarily a closed set C and a point $x \notin C$. For every $c \in C$, there exists a finite separator S_c separating c from x, that is, such that $G \setminus S_c$ is the union of two disjoint open sets $A(c)$, $B(c)$ containing c and x respectively. Since S_c is closed, $A(c)$, $B(c)$ are open in X. Since C is a closed subset of the compact

[2] In [2] these spaces are called *regular*, except that these authors reserve this term for compact metric spaces. However, for general topological spaces the term *regular* sometimes has another meaning, according to which all metric spaces are regular.

space M, C is compact and therefore the union A of $A(c)$ over some finite subset I of C covers C. But the intersection B of the $B(c)$ over $c \in I$ is also open. Clearly A, B are disjoint and contain C and x respectively. Moreover, the complement of $A \cup B$ is contained in the union of the finite S_c over $c \in I$ and is therefore the required finite separator of C and x.

Theorem 2.5. *Let G be a graph-like metric continuum. Then G is finitely separated.*

Proof. Let y_1, y_2 be arbitrary points of G. If either is contained in an edge, it is easily seen to be separated from the other by two points. So we assume that both are vertices.

Since V is zero-dimensional, there exists a partition of V into disjoint sets V_1, V_2, closed in V and therefore in G, and containing y_1, y_2 respectively. As G is compact, there exist disjoint open sets U_1, U_2 in G containing V_1, V_2 respectively. Let us say that an edge is *covered* if it is contained in $U_1 \cup U_2$, and *uncovered* otherwise. We observe that there are only finitely many uncovered edges, for if this were not the case, we could find an infinite sequence of points outside $U_1 \cup U_2$, each from some uncovered edge but no two from the same edge, converging to some point v which necessarily must be an vertex, contradicting the fact that the open set $U_1 \cup U_2$ contains all of V.

Let us now distinguish edges into *cross-edges*, that is, those with one endvertex in each of V_1, V_2, and *i-edges*, those with both endvertices in V_i, for $i = 1, 2$. Note that cross-edges and i-edges necessarily intersect U_i. Moreover, any edge e intersecting both U_1 and U_2, in particular all cross-edges, must be uncovered, for otherwise the disjoint open sets U_1, U_2 give a separation of e, contradicting the fact that e is connected. Therefore, by deleting (if necessary) from U_i the finitely many $(3-i)$-edges intersecting U_i, and adding to U_i the uncovered i-edges, we may assume that all i-edges are contained in U_i.

Finally, for each cross-edge e, we select a point $s(e) \in e \setminus (U_1 \cup U_2)$. Then $e \setminus s(e)$ consists of two components e_1, e_2, which are open in G. We add e_i to U_i (and delete it from U_{3-i}) if and only if the endvertex of e in e_i lies in U_i. Then the U_i are still open, disjoint and respectively contain x_i; thus, the set of points $s(e)$, over the finitely many (uncovered) cross-edges e, is the required separator of y_1 and y_2. ∎

We remark that finitely separated continua are known to be HLC; see Blumenthal and Menger [2]. However, the proof of this theorem is non-trivial.

3. Augmenting arcs and the disjoint-arc version of Menger's Theorem

We are now ready for the main results, namely an adaptation of the augmenting path method for (most of) the metric spaces discussed in the previous sections.

We begin with the totally disjoint case which is easy, even in the non-metric case. Then we establish a fan-version (that is, the arcs have distinct starting points but the same terminating point). Finally, we prove the general version, which is for internally disjoint arcs.

Let A,B be disjoint sets in a Hausdorff space M. An A–B *arc* is an arc from A to B having only its first and last point in common with $A \cup B$.

Let $P_1, P_2, \ldots, P_k$ be pairwise disjoint A–B arcs. We define an *augmenting arc* Q from A as an arc (not necessarily a simple arc) which satisfies the following. Its first point is in A and not in $P_1 \cup P_2 \cup \cdots \cup P_k$. It is the union of a finite collection $Q_1, Q_2, \ldots, Q_m$ of simple arcs (numbered so that consecutive arcs share a common endpoint). Q_1 is an arc from A to $P_1 \cup P_2 \cup \cdots \cup P_k$ having only its first point in common with A and only its last point in common with $P_1 \cup P_2 \cup \cdots \cup P_k$. For each odd i, $1 < i < m$, Q_i is a simple arc from $P_1 \cup P_2 \cup \cdots \cup P_k$ to $P_1 \cup P_2 \cup \cdots \cup P_k$ having only its first and last point in common with $A \cup B \cup P_1 \cup P_2 \cup \cdots \cup P_k$. If this Q_i starts and ends at the same P_j, then the last point of Q_i succeeds the first point of Q_i. (This condition is usually not imposed as it is not needed for finite graphs. It becomes useful when we later allow an augmenting arc to consist of an infinite collection $Q_1, Q_2, \ldots$.) Finally, for each even i, Q_i is a backward arc in some P_j. If m is odd, then Q_m, the last arc in the sequence $Q_1, Q_2, \ldots, Q_m$, is a simple arc from $P_1 \cup P_2 \cup \cdots \cup P_k$ to $B \cup P_1 \cup P_2 \cup \cdots \cup P_k$ having only its first point in common with $A \cup P_1 \cup P_2 \cup \cdots \cup P_k$ and only its last point in common with $B \cup P_1 \cup P_2 \cup \cdots \cup P_k$. We also require that the last point of Q_m is not in any of $Q_1, Q_2, \ldots, Q_{m-1}$. Any two of the arcs $Q_1, Q_2, \ldots, Q_m$ are disjoint except that a point of $P_1, P_2, \ldots, P_k$ may be a common end of some Q_i and Q_j, even when $|i-j| > 1$. See Figure 4. Hence an augmenting arc has only finitely many self-intersections. So, if i and j are odd, then Q_i and Q_j cannot have a common terminating point. For that would imply that Q_{i+1} and Q_{j+1} have an arc in common, and this is not allowed.

For convenience, we add an additional condition: If i,j are even numbers, $i < j$, and Q_i, Q_j are backward arcs on the same P_q, then Q_j succeeds Q_i on P_q. (Also this condition is usually not imposed as it is not needed for finite graphs.)

An *augmenting semi-arc* is defined analogously except that now either Q_m is a semi-arc or else the sequence $Q_1, Q_2, \ldots$ is infinite.

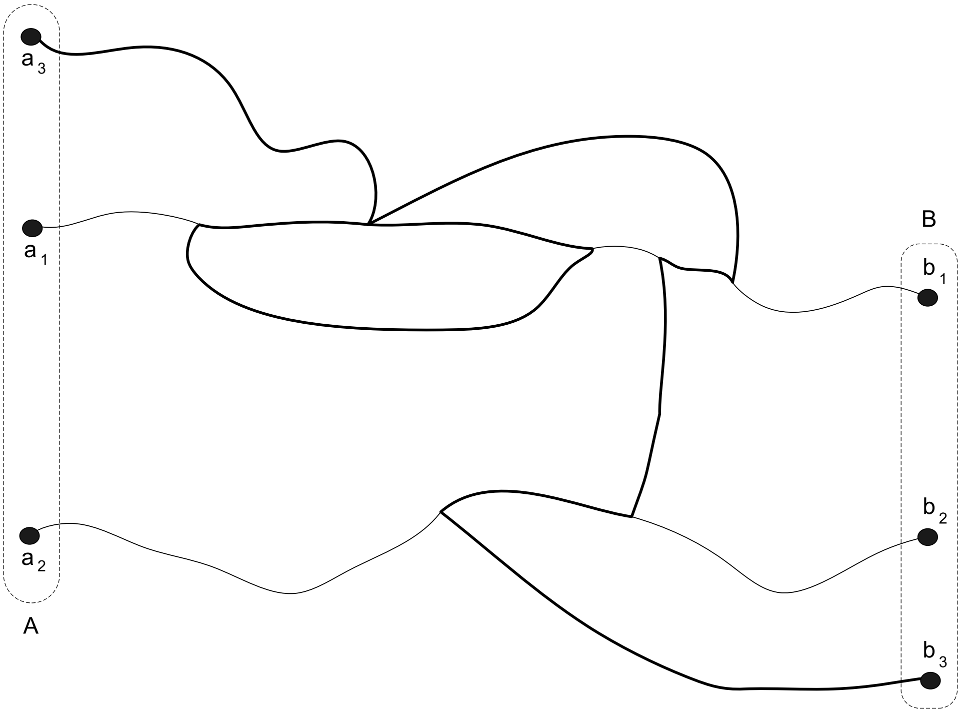


Figure 4. Augmenting arc

If Q_m ends at a point in B but not in any of the P_i, then $Q \cup P_1 \cup P_2 \cup \cdots \cup P_k$ contains a collection of $k+1$ pairwise disjoint A–B arcs. To see this, we consider $Q \cup P_1 \cup P_2 \cup \cdots \cup P_k$ as a graph, and we delete those edges which are contained in both Q and $P_1 \cup P_2 \cup \cdots \cup P_k$. In the resulting graph, which we call G, every vertex in A or B has degree 1, and all other vertices have degree 2. Hence every component of G is a path or a cycle.

Consider a component of G intersecting A. That component is clearly a path, P say, in G (and hence a simple arc in M), and when we follow this path P from A we never return to A, because then, just before we return to A, we use an edge both in Q and $P_1 \cup P_2 \cup \cdots \cup P_k$, a contradiction. Hence we terminate at B. This observation leads to the disjoint-arc version of Menger's Theorem.

Theorem 3.1. *Let A, B be disjoint sets in a Hausdorff space M, and let k be a natural number. Assume that, for each set S with at most k elements, $M \setminus S$ has an A–B arc. Then M has a collection of $k+1$ pairwise disjoint A–B arcs.*

Proof of Theorem 3.1. The proof is by induction on k. For $k=0$ there is nothing to prove. Assume now that $P_1, P_2, \dots, P_k$ are pairwise disjoint A–B arcs. If we delete all starting points, there still is an A–B arc. So, there exists an augmenting arc from A to $P_1 \cup P_2 \cup \dots \cup P_k$. We may think of P_i as the image of a function f_i from $[0,1]$ to M. If S is a subset of P_i, then the supremum of S is defined as $f_i(a)$ where a is the supremum of $f_i^{-1}(S)$. If S is empty, then its supremum is $f_i(0)$. For each $i = 1, 2, \dots, k$, let t_i be the supremum (on P_i) of the terminating points of augmenting arcs starting at A. By the assumption of the Theorem, $M \backslash \{t_1, t_2, \dots, t_k\}$ contains an A–B arc Q'. Then Q' contains a subarc Q which starts in A or a predecessor of some t_i on P_i and ends in B or in a successor of some t_i on P_i and which has only its first and last point in common with $A \cup B \cup P_1 \cup P_2 \cup \dots \cup P_k$. Then Q must end in a point in B which is not in $P_1 \cup P_2 \cup \dots \cup P_k$. For, if Q ends in a point t'_j which is successor of t_j on P_j, then Q cannot start in A because then Q would be augmenting, contradicting the definition of the supremum t_j. On the other hand, if Q starts in a predecessor t'_i of t_i on P_i, then we consider an augmenting arc Q'' which ends at a point on P_i between t'_i and t_i. Now $Q \cup Q''$ together with a segment of P_i contains an augmenting arc ending at t'_j, again a contradiction to the definition of t_j. So, we may assume that Q ends at a point in B and not in $P_1 \cup P_2 \cup \dots \cup P_k$. If Q starts in a predecessor t'_i of t_i on P_i, then we define Q'' as above, and then $Q \cup Q''$ together with a segment of P_i contains an augmenting arc from A to B. Now the observation preceding Theorem 3.1 gives the desired $k+1$ pairwise disjoint A–B arcs.

This completes the proof of Theorem 3.1. ∎

Note that if we assume that A, B are closed subsets of M, then the condition that $M \backslash S$ has an A–B arc may be replaced by the weaker condition that $M \setminus S$ has a simple arc starting in A and terminating at B, since any such arc contains an A–B arc. Also note that A and B may intersect if we modify the definition of an arc to allow it to consist of just one point.

4. Menger's Theorem

While Menger's Theorem in the totally disjoint version is easy and holds for general Hausdorff spaces, the examples of Figures 1, 2 show that additional conditions are needed for the stronger versions. We choose in this section the property of being locally arcwise connected and strongly HLC and metric. However, all arguments can be modified to cover also the locally arcwise connected, HLC metric spaces. It is not clear if such modifications are worth the efforts since the two properties are equivalent for many spaces, as shown by Wilder [39] and Tymchatyn [34].

Lemma 4.1. *Let s_1, s_2, t be distinct elements in a locally arcwise connected, strongly HLC metric space M. Let P be a simple arc from s_1 to t which does not contain s_2. Assume that, for each $\epsilon > 0$, M has a simple arc disjoint from P which starts in s_2 and terminates in an element of distance less than ϵ from t. Then M has a simple arc from s_2 to t which is disjoint from $P \backslash \{t\}$.*

Proof of Lemma 4.1. We construct a sequence of simple arcs $P_1, P_2, \ldots$ disjoint from P and with the following properties: P_1 starts in s_2 and terminates in an element of distance less than 1 from t. For each natural number $n > 1$, P_n terminates in an element of distance less than $1/n$ from t, starts in an element of $P_1 \cup P_2 \cup \cdots \cup P_{n-1}$ and has no other element in common with $P_1 \cup P_2 \cup \cdots \cup P_{n-1}$. We shall add an additional condition which we shall refer to as the *neighbourhood condition*: For each element x of $P_n \backslash (P_1 \cup P_2 \cup \cdots \cup P_{n-1})$, we let $U(x)$ be an open, arcwise connected neighbourhood of x such that $U(x)$ is at a positive distance from t, and has diameter less than $1/n$, and let $U'(x)$ be an open neighbourhood of x such that the closure of $U'(x)$ is contained in $U(x)$. As P_n is compact it is covered by finitely many neighbourhoods of the form $U'(x)$ (where x belongs to $P_1 \cup P_2 \cup \cdots \cup P_n$), and we ignore the others whenever we speak of the sets of the form $U(x)$ or $U'(x)$.

When we add P_n to $P_1 \cup P_2 \cup \cdots \cup P_{n-1}$ we make sure that the last point of P_n is sufficiently close to t so as to be outside the closure of W_{n-1}, which we define to be the union of $U(y)$ over all y in $P_1 \cup P_2 \cup \cdots \cup P_{n-1}$. We follow P_n backwards; we stop the first time we hit the closure of some $U'(x)$ and we use the arcwise connected neighbourhood $U(x)$ to continue. More precisely, P_n has the property that there exists an element x such that the intersection of P_n with the union of all $U'(y)$, taken over all y in $P_1 \cup P_2 \cup \cdots \cup P_{n-1}$, is contained in an arc that is entirely in $U(x)$. Moreover, when choosing $U(x)$ for $x \in P_n \setminus (P_1 \cup P_2 \cup \cdots \cup P_{n-1})$, we make sure that, if x lies inside W_{n-1}, or outside $\mathtt{cl}(W_{n-1})$, then so does $U(x)$; note this implies that if x is on the boundary of W_{n-1}, then of course x is in the interior of $U(x)$ (by the definition of $U(x)$), but x is not in the interior of $U(y)$ for any y distinct from x.

Claim. For fixed n, only finitely many P_j can intersect $\mathtt{cl}(W_n)$.

Proof of Claim. Suppose, by way of contradiction, that for infinitely many $j > n$, P_j intersects $\mathtt{cl}(W_n)$, and let, for any such j, P'_j be the subarc intersecting $\mathtt{cl}(W_n)$ in precisely one point, which we denote by p_j, and ending at the same point as P_j. We consider the set Q consisting of the union of $\mathtt{cl}(W_n)$

with all the P'_j; since W_n is connected, so are $\mathtt{cl}(W_n)$, Q and $Q^\star := Q \cup \{t\}$. Hence $Q^\star$ is locally connected at t because M is strongly HLC.

Since the diameter of $U(p_j)$ tends to zero, we may choose an open neighbourhood V of t in $Q^\star$ disjoint from (the closure of) W_n and all the $U(p_j)$. Let V' be a connected neighbourhood of t contained in V, and let k be the smallest index so that P'_k intersects V'. Since P'_k is closed, $P'_k \cap V'$ is closed in V'. We claim it is also open in V'.

To see this, we choose arbitrarily $z \in P'_k \cap V'$. Then z belongs to some $U'(y)$, for some $y \in P_k$. By the choice of V, $y \notin \mathtt{cl}(W_n)$; in fact all of $U(y)$ is disjoint from $\mathtt{cl}(W_n)$. By the definition of P'_j, no P'_j with $j > k$ meets $U'(y)$. Hence $U'(y)$ is disjoint from $P'_j \cap V'$ for $j > k$. Recall $U'(y)$ does not contain t. Hence $P'_k \cap V'$ is closed and open in V', a contradiction which proves the claim. ∎

Since each arc P_i is contained in some W_n, the claim implies that every arc P_i intersects only finitely many P_j. Put $G := P_1 \cup P_2 \cup \cdots$. Then we may think of G as a locally finite graph (in fact a tree) whose vertices of degree at least 2 are the points contained in at least two P_i. A vertex of degree 1 in G is an endpoint of some P_i which is not contained in any other P_j. Furthermore, if w is a point (not necessarily vertex) in G, then for some n, W_n is a neighbourhood of w disjoint from all but finitely many edges; this neighbourhood can be modified to a smaller neighbourhood, say $A(w)$, which is disjoint from *all* the edges whose closure does not contain w: Just intersect W_n with the (finitely many) complements of closures of edges not containing w.

By Kőnig's Infinity Lemma, G contains an infinite path P. We shall prove that $P \cup \{t\}$ is the desired arc. We think of $P \cup \{t\}$ as the image of a map f from $[0,1]$ to M where $f(1) = t$. It suffices to prove continuity of f at 1. To prove this, it is sufficient to show that, for every neighbourhood N of t, P eventually enters N and stays in N.

Consider first the case where $t \notin \mathtt{cl}(P)$. Then we consider a neighbourhood N of t disjoint from P. In G we can find pairwise disjoint paths $Q_1, Q_2, \ldots$ such that each Q_i has an end t_i in N and the other end in P. Moreover, we may assume that t_i tends to t as i tends to infinity. Then $M' = \{t\} \cup P \cup Q_1 \cup Q_2 \cup \cdots$ is connected. As M is HLC, M' has an open connected neighbourhood N' around t and contained in N. Choose i such that Q_i intersects N'. Clearly, $Q_i \cap N'$ is closed in N' because Q_i is closed. The union of the neighbourhoods $A(w)$, taken over all w in Q_i show that $Q_i \cap N'$ is also open in N', a contradiction to the assumption that N' is connected.

So we assume that $t \in \mathtt{cl}(P)$, and hence $P \cup \{t\}$ is connected. Again, we consider an arbitrary neighbourhood N of t, and we shall show that P eventually enters N and stays in N. Suppose therefore (reduction ad absurdum) that this is not so. $P\cup\{t\}$ has an open connected neighbourhood U around t and contained in N. Extend U to an open connected neighbourhood N' of t in M such that U is the intersection of N' with $P\cup\{t\}$ and N' is contained in N. Then P enters and leaves N' infinitely often. Let Q be an arcwise connected component of $N'\cap P$. Then Q is of the form $f^{-1}((a,b))$. As $Q=N'\cap f([a,b])$, Q is closed in N'. We shall prove that Q is open too, and thereby reach a contradiction. Choose c,d in $[0,1]$ such that $c<a<b<d$ and such that $f(c),f(d)$ are vertices of G. The union B of the neighbourhoods $A(w)$ taken over all w in Q is disjoint from $f([0,c])\cup f([d,1])$. As the restriction of f to $[c,d]$ is a homeomorphism of $[c,d]$ onto $f([c,d])$, it follows that Q is open in $B\cap P$ and hence also in $N'\cap P$. This contradiction completes the proof. ∎

Lemma 4.2. *Let s_1,s_2,t be distinct elements in a locally arcwise connected, strongly HLC metric space M. Assume that for every element x distinct from t, $M\setminus\{x\}$ has a simple arc from $\{s_1,s_2\}$ to t. Then M has two simple arcs from $\{s_1,s_2\}$ to t which have only t in common.*

Proof of Lemma 4.2. Let P be a simple arc from s_1 to t not containing s_2. We now construct (if possible) a sequence $A_1,A_2,\dots$ of augmenting arcs starting at s_2 as follows: A_1 is a simple arc from s_2 to a point a_1 on P. Then A_2 consists of A_1 followed by a backwards arc B_1 from a_1 to b_1 on P and then followed by a simple arc A_2' from b_1 to a_2, such that a_2 is a successor of a_1 on P and such that A_2' has only b_1,a_2 in common with $P\cup A_1$. And A_3 consists of A_2 followed by a backwards arc B_2 from a_2 to b_2 on P followed by a simple arc A_3' from b_2 to a_3 such that a_3 is a successor of a_2 on P and such that A_3' has only b_2,a_3 in common with $P\cup A_2$. Moreover, either $b_2=a_1$ or b_2 is a successor of a_1 on P. The arcs $A_3,A_4,\dots$ are defined analogously.

We shall now give a precise definition of this infinite sequence $A_1,A_2,\dots$ of augmenting arcs (or complete the proof of Lemma 4.2). We first construct A_1. If we follow a simple arc A from s_2 to t there will be a first time that A meets P because P is compact. We call this point the *hitting point* of A on P. Let a_1 be the supremum of the set of hitting points on P of all simple arcs from s_2 to t. If we apply Lemma 4.1 to $P[s_1,a_1]$, then we conclude that a_1 itself is a hitting point of some simple arc which we call A_1. If $a_1=t$, the proof is complete, so assume that this is not the case.

Now $M\setminus\{a_1\}$ has a simple arc from $\{s_1,s_2\}$ to t. This arc must intersect $P[s_1,a_1]$, for otherwise we get a contradiction to the assumption that a_1 is a supremum. Hence $M\setminus\{a_1\}$ has a simple arc from $\{s_1\}$ to t. Let a_2 be the supremum of the set of hitting points on $P[a_1,t]$ of all simple arcs in $M\setminus\{a_1\}$ from s_1 to t. If we apply Lemma 4.1 to $P[a_1,a_2]$, then we conclude that a_2 itself is a hitting point of such a simple arc. We follow this arc backwards from a_2 until we hit $P[s_1,a_1]$ in a point b_1, say. (Note that we do not hit A_1 because of the definition of a_1.) This new arc from b_1 to a_2 is denoted A'_2, and the backward arc on P from a_1 to b_1 is denoted B_1. Let a_3 be the supremum of the set of hitting points on $P[a_2,t]$ of all simple arcs in $M\setminus\{a_2\}$ from s_1 to t. Again, we conclude that a_3 itself is a hitting point of such a simple arc. We follow this arc backwards from a_3 until we hit $P[s_1,a_2]$ in a point b_2, say. This new arc from b_2 to a_3 is denoted A'_3, and the backward arc on P from a_2 to b_2 is denoted B_2. The definition of a_1,a_2 implies that b_2 is in $P[a_1,a_2)$. Note that possibly $b_2=a_1$. Also note that A'_3 is disjoint from A'_2 and $A_1\setminus\{a_1\}$.

We continue defining $A_1,A_2,\dots$ in this way. If $a_n=t$ for some n, then we terminate the sequence $A_1,A_2,\dots$. But then there exists an augmenting arc from s_2 to t, and now the argument of Theorem 3.1 completes the proof of Lemma 4.2.

Suppose therefore that each a_n is distinct from t. We claim that a_n tends to t as n tends to infinity. For suppose (*reductio ad absurdum*) that a_n tends to t' as n tends to infinity, where $t'\neq t$. Now $M\setminus\{t'\}$ has a simple arc from $\{s_1,s_2\}$ to t. This arc has a hitting point, t'' say, on $P[t',t]$. We follow this arc backwards from t'' until we hit $P[s_1,t']$ in a point c, say, and we denote this new arc from c to t'' by C. (Note that the definition of a_1 shows that C does not hit A_1, the definition of a_2 shows that C does not hit A'_2, etc.) As a_n tends to t' as n tends to infinity, there will be a smallest n such that a_n is in $P(c,t')$. But now C contradicts the definition of the supremum a_{n+1}. This proves the claim that a_n tends to t as n tends to infinity.

The union of the arcs $A_1,A_2,\dots$ is an augmenting semi-arc. The observation preceding Theorem 3.1 shows that the union of P and this augmenting semi-arc contains two disjoint simple semi-arcs Q_1,Q_2. They start at s_1,s_2, respectively. The simple semi-arc Q_1 contains all a_n with n even, and Q_2 contains all a_n with n odd. So each of Q_1,Q_2 has points arbitrarily close to t. We now complete the proof by showing that each of Q_1,Q_2 converges to t. It suffices to prove this for Q_1. (The argument for Q_2 is similar.) Clearly, Q_1 is arcwise connected. Let Q be the union of Q_1 and t. Then Q is connected and therefore locally connected at t. Consider any neighbourhood of t in M. We shall prove that from a certain time and onwards, Q_1 is entirely inside this neighbourhood. The neighbourhood contains an open neighbourhood W around t such that $W\cap Q$ is connected.

We shall show that Q_1 must be inside W from a certain time and onwards. For if Q_1 enters W and later has a point x outside W, we let Q'_1 denote the intersection of W with the closed set $Q_1[s_1,x]$. Clearly, Q_1' is closed in $W \cap Q$. We claim that it is also open, contradicting the fact that $W \cap Q$ is connected. To see this, observe first that for any point w belonging to any A_i, there clearly exists in M an open, arcwise connected neighbourhood disjoint from all B_j with $j > i$. Moreover, this neighbourhood contains an open, arcwise connected neighbourhood $B(w)$ disjoint from all A_j' with $j > i$; for otherwise, any arcwise connected neighbourhood of w must intersect A_j' for arbitrarily high j, and can be used to contradict the maximal choice of a_i. This applies in particular to the points w inside W. If Q_1' were not open in $Q \cap W$, there would be some point $p \in Q_1'$ such that for every neighbourhood Z of p, infinitely many A_i' intersect Z. But then the neighbourhood $B(p)$ shows that this is not possible.

This completes the proof of Lemma 4.2. ∎

Lemma 4.3. *Let $t, s_1, s_2, \dots, s_k$ be distinct elements in a locally arcwise connected, strongly HLC metric space M. Assume that for every set S consisting of less than k elements, all distinct from t, $M \setminus S$ has a simple arc from $\{s_1, s_2, \dots, s_k\}$ to t. Then M has k simple arcs from $\{s_1, s_2, \dots, s_k\}$ to t which have only t in common, pair by pair.*

Proof. The k simple arcs in Lemma 4.3 are called a *k-fan* from $\{s_1, s_2, \dots, s_k\}$ to t. The proof is by induction on k. For $k = 1$, there is nothing to prove. The case $k = 2$ is proved by Lemma 4.2. We now assume that the statement of Lemma 4.3 is true and proceed to the case of a $(k+1)$-fan from a set $\{s_1, s_2, \dots, s_{k+1}\}$ to t. Using the induction hypothesis we may assume that M contains a k-fan $Q_1, Q_2, \dots, Q_k$ from $\{s_1, s_2, \dots, s_k\}$ to t. By permuting the points of $\{s_1, s_2, \dots, s_{k+1}\}$, if necessary, we may assume that s_{k+1} is not contained in any of $Q_1, Q_2, \dots, Q_k$. We now repeat the idea in the proof of Lemma 4.2. Instead of defining an increasing sequence of augmenting arcs $A_1, A_2, \dots$, we define an increasing sequence $G_1, G_2, \dots$ of finite graphs with the following properties. If G_j does not intersect Q_i, then we put $a_{i,j} = s_i$. If G_j intersects Q_i, then we denote by $a_{i,j}$ the maximum point of G_j on Q_i. In this case G_j contains $Q_i[s_i, a_{i,j}]$, and an augmenting arc from s_{k+1} to $a_{i,j}$ (and hence to every point preceding $a_{i,j}$ on Q_i) for each $i = 1, 2, \dots, k$ and for each $j = 1, 2, \dots$. In order to define G_1, we define an increasing sequence $H_{1,1}, H_{2,1}, \dots, H_{k,1}$ of finite graphs. If there exists a simple arc from s_{k+1} to Q_1 disjoint from each of $Q_2, Q_3, \dots, Q_k$, then we let $a_{1,1}$ be the supremum on Q_1 of the hitting points of such paths, and the proof of Lemma 4.1 shows that this supremum is attained, that is, M has a simple arc $A_{1,1}$ say, from s_{k+1} to Q_1 disjoint from each of $Q_2, Q_3, \dots$ hitting Q_1 at $a_{1,1}$. We may assume that $a_{1,1}$ is distinct from t since otherwise the proof is complete. Now $H_{1,1}$ is defined as the union of $A_{1,1}$ and $Q_1[s_1,a_{1,1}]$. If $A_{1,1}$ does not exist, then $H_{1,1}$ consists of s_{k+1} only. If there exists a simple arc from $H_{1,1}$ to Q_2 disjoint from each of $Q_1[a_{1,1},t],Q_3,\dots,Q_k$, then we let $a_{2,1}$ be the supremum on Q_2 of the hitting points of such paths, and the proof of Lemma 4.1 shows that this supremum is attained, that is, M has a simple arc $A_{2,1}$ say, from $H_{1,1}$ to Q_2 disjoint from each of $Q_1[a_{1,1},t],Q_3,Q_4,\dots$ hitting Q_2 at $a_{2,1}$. Now $H_{2,1}$ is defined as the union of $H_{1,1}$, $A_{2,1}$ and $Q_2[s_2,a_{2,1}]$. If $A_{2,1}$ does not exist, then we put $H_{2,1}=H_{1,1}$. More generally, $H_{i+1,1}$ is defined as $H_{i,1}\cup A_{i+1,1}\cup Q_{i+1}[s_{i+1},a_{i+1,1}]$, where $A_{i+1,1}$ is a simple arc from $H_{i,1}$ (minus the suprema) to Q_{i+1}, or else $H_{i+1,1}=H_{i,1}$ if the arc $A_{i+1,1}$ does not exist. When $H_{k,1}$ has been defined, we put $G_1=H_{k,1}$.

When we extend G_j to G_{j+1} we first delete $a_{1,j},a_{2,j},\dots,a_{k,j}$, and then we successively add, if possible, a simple arc $A_{i,j+1}$, $i=1,2,\dots,k$. The arc $A_{1,j+1}$ terminates on Q_1 in a point denoted by $a_{1,j+1}$ succeeding $a_{1,j}$ and starts in a point of G_j. Again, $a_{1,j+1}$ is the supremum of the possible hitting points. We define $H_{1,j+1}$ as the union of G_j, $A_{1,j+1}$ and the arc $Q_1[s_1,a_{1,j+1}]$. We continue like this defining $H_{2,j+1},H_{3,j+1},\dots,H_{k,j+1}$, and we put $G_{j+1}=H_{k,j+1}$. Clearly, each G_n may be thought of as a finite graph. But also the union $G_1\cup G_2\cup\cdots$ is a graph. An edge of G_n may be subdivided into edges in some G_m with $m>n$. But this can happen only finitely often because of the definition of the suprema.

If some $a_{i,j+1}$ equals t, then we stop the first time this happens, and we obtain the desired $(k+1)$-fan as in the proof of Theorem 3.1. So assume that all these suprema are distinct from t. For each fixed i, the sequence $a_{i,1},a_{i,2},\dots$ converges to a_i, say. If each of $a_1,a_2,\dots,a_k$ is distinct from t, then we delete them and consider an arc from $\{s_1,s_2,\dots,s_{k+1}\}$ to t. That arc contains a subarc from a point preceding one of $\{a_1,a_2,\dots,a_k\}$ to a point succeeding one of $\{a_1,a_2,\dots,a_k\}$. That arc contradicts the definition of the suprema. So we may choose the notation such that $a_1=t$. It follows that, for each natural number n, there is a natural number i such that G_i contains an augmenting arc R_n from s_{k+1} to a point on Q_1 of distance less than $1/n$ to t. Following R_n backwards, we may assume that when it hits some R_m with $m<n$, then R_m,R_n agree from there all the way back to s_{k+1}. We now apply an adaption of Kőnig's Infinity Lemma to the union $R_1\cup R_2\cup\cdots$ as in the proof of Lemma 4.1. As in the proof of Lemma 4.1 we conclude that $R_1\cup R_2\cup\cdots$ contains an augmenting semi-arc R, say, such that R either has t as an accumulation point, or else there are pairwise disjoint simple arcs starting at R and ending at points converging towards t. As in the proof of Lemma 4.1 we obtain a contradiction unless the augmenting semi-arc R has t as an accumulation point. As in the proof of Lemma 4.2 we then

conclude that R in fact converges towards t. For this, we define at the end of Lemma 4.2 a neighbourhood which we called $B(w)$. We do the same in the present proof except that we encounter a problem with this definition when w is of the form a_i where $a_i \neq t$. So when we obtain the contradiction as in the proof of Lemma 4.2, we make sure that the set which we prove to be both closed and open (the set which in the proof Lemma 4.2 is called Q'_1) avoids the finitely many points of the form a_i where $a_i \neq t$.

Then we apply the observation preceding Theorem 3.1 to the arcs $Q_1, Q_2, \dots, Q_k$ and the augmenting semi-arc R. This observation shows that $R \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k$ contains a collection of $k+1$ simple semi-arcs which form a $(k+1)$-fan except that the $k+1$ semi-arcs, which we call $Q'_1, Q'_2, \dots, Q'_{k+1}$, do not necessarily converge to t. If Q'_i contains only finitely many arcs which are outside $Q_1 \cup Q_2 \cup \cdots \cup Q_k$, then Q'_i eventually stays inside some Q_j and follows therefore that Q_j till it reaches t. On the other hand, it is possible that Q'_i contains infinitely many arcs which are outside $Q_1 \cup Q_2 \cup \cdots \cup Q_k$, and we need an argument to ensure that Q'_i converges towards t. Let N be any open neighbourhood around t. Any arc R' in Q'_i which is outside $Q_1 \cup Q_2 \cup \cdots \cup Q_k$ is also an arc of R. (This follows from the way in which Q'_i is constructed.) And all those arcs R' (except finitely many) are inside N. Any arc R'' in Q'_i which is inside $Q_1 \cup Q_2 \cup \cdots \cup Q_k$ is a forward arc on some Q_j. Any such arc R'' succeeds on Q_j a backward arc in R. Those backward arcs will eventually be inside N (because R converges towards t), and so will the arcs of the form R'' (because Q_j converges towards t). Hence each Q'_i converges to t.

This completes the proof of Lemma 4.3. ∎

Theorem 4.4. *Let s,t be distinct elements in a locally arcwise connected, strongly HLC metric space M. Assume that for every set S consisting of at most k elements, all distinct from s,t, $M \setminus S$ has a simple arc from s to t. Then M has $k+1$ simple arcs from s to t which have only s,t in common, pair by pair.*

Proof of Theorem 4.4. The proof is by induction on k. For $k=0$, there is nothing to prove. Assume now, by the induction hypothesis that M has k simple arcs $P_1, P_2, \dots, P_k$ from s to t which have only s,t in common, pair by pair. We shall show the existence of $k+1$ such arcs. Select a point $a_{i,1}$ distinct from s,t on P_i for each $i = 1,2,\dots,k$. Ignoring $P_i(s,a_i)$ for each $i = 1,2,\dots,k$, the proof of Lemma 4.3 shows that M has an augmenting semi-arc from s which converges to t. We follow that semi-arc backwards until it hits $P_1[s,a_{1,1}) \cup P_2[s,a_{2,1}) \cup P_k[s,a_{k,1})$ and call the resulting semi-arc S_1. If S_1 starts on P_1, say, then there may be several possible choices for the starting point of S_1. We let S_1 start in a point of distance at most 1 from the infimum of the possible starting points. (Here it is not completely clear that the infimum can be attained, as S_1 is not just an arc. It is also an augmenting arc.)

Now define $a_{i,2}$ as the smallest point of S_1 on P_i for $i = 1,2,\dots,k$. If S_1 does not intersect P_i, then we put $a_{i,2} = a_{i,1}$. Now we repeat the argument above to find an augmenting semi-arc S_2 from a point on $P_1[s,a_{1,2}) \cup P_2[s,a_{2,2}) \cup \cdots \cup P_k[s,a_{k,2})$. If S_2 hits S_1 in such a way that we can continue along S_1 towards t, then we do so. (Note that there may be some special hitting points which will not allow us to do so, namely endvertices of the maximal subarcs of S_1 which are backward on some P_i.) Also, if S_2 hits some P_i in a point above some point of S_1, then we follow that P_i backwards until we hit S_1, and then we continue along S_1. We choose S_2 such that it starts on P_i, where i is the first possible number in the sequence $2,3,\dots,1$. More generally, if S_{n-1} starts on P_j, we choose S_n such that it starts on P_i, where i is the first possible number in the sequence $j+1, j+2,\dots,k,1,2,\dots,j$. On that P_i we choose the starting point of S_n such that it has distance at most $1/n$ from the infimum of the possible starting points. Continuing in this manner, as in the proof of Lemma 4.3 we conclude that for some i $(i=1,2,\dots,k)$, say $i=1$, $a_{1,n}$ tends to s as n tends to infinity. Now $S_1 \cup S_2 \cup \cdots$ contains an augmenting semi-arc from $a_{1,n}$ to t. Let this arc be called S'_n. We may assume that the sequence $S'_1, S'_2, \dots$ is chosen such that, when S'_n hits $S'_1 \cup S'_2 \cup \cdots \cup S'_{n-1}$, then it coincides with some S'_i $(i<n)$ thereafter. Clearly, $S'_1 \cup S'_2 \cup \cdots \cup S'_n$ is a locally finite graph. The same is the case for $S'_1 \cup S'_2 \cup \cdots$. For otherwise $S'_1 \cup S'_2 \cup \cdots$ contains an infinite fan or some $S'_1 \cup S'_2 \cup \cdots \cup S'_n$ contains an edge which is subdivided infinitely often when we add $S'_{n+1}, S'_{n+2}, \dots$, and we obtain a contradiction as in the proof of Lemma 4.1. By the argument of Lemma 4.1 (where we used Kőnig's Infinity Lemma), we may assume that $S'_1 \cup S'_2 \cup \cdots$ contains an augmenting semi-arc converging towards s. Clearly, $S'_1 \cup S'_2 \cup \cdots$ also contains an augmenting semi-arc converging towards t. We may assume that these two semi-arcs have precisely their starting points in common, and we call their union A. Now we claim that the symmetric difference of A and $P_1 \cup P_2 \cup \cdots \cup P_k$, which we call B, constitutes the desired $k+1$ arcs from s to t.

To prove this claim we first observe that each arcwise connected component of $B \setminus \{s,t\}$ together with s,t is a simple arc from s to t. (The definition on an augmenting arc ensures that it cannot both start and end in s, say.)

The only question remains how many s–t arcs B contains. Let $D_s(1/n)$ and $D_t(1/n)$ denote the closed discs of radius $1/n$ around s and t, respectively. Let $A_n, P_{i,n}$ be subarcs of A, P_i respectively, $i = 1, 2, \ldots, k$ such that they have only the first and last point in common with $D_s(1/n) \cup D_t(1/n)$. Then the symmetric difference of A_n and $P_{1,n} \cup P_{2,n} \cup \cdots \cup P_{k,n}$ consists of $k+1$ pairwise disjoint arcs. Hence B contains $k+1$ pairwise disjoint arcs each of which starts arbitrarily close to s and ends arbitrarily close to t. A set which is the union of at most k internally disjoint arcs from s to t does not have that property (as follows easily from continuity). Hence B is the desired collection of $k+1$ internally disjoint arcs from s to t.

This completes the proof of Theorem 4.4. ∎

5. Menger's Theorem for other spaces

In this section we point out that the augmenting arc method used in the previous section also applies, with minor adjustments, to other spaces as well. We begin with the afore-mentioned result of Whyburn [38].

Theorem 5.1. *Let s, t be distinct points in a locally compact, locally connected, metric space M, and let k be a natural number. Assume that, for every set S consisting of at most k elements, all distinct from s, t, $M \setminus S$ has a simple arc from s to t. Then M has $k+1$ simple arcs from s to t which have only s, t in common, pair by pair.*

Proof of Theorem 5.1. As M is locally compact and locally connected, M is also locally arcwise connected. We can therefore repeat the proofs of Lemma 4.1, Lemma 4.2, and Theorem 4.4 word for word except that we cannot use the property HLC. When we built a subspace consisting of arcs which eventually gave the desired simple arc converging to t, we used the property HLC which we cannot use now. Therefore, we add an additional minimality condition on the added arcs, namely that the diameter of the added arc is as small as possible. As this minimum diameter can perhaps not be achieved, we say instead that the diameter is at most twice as much as the infimum of the possible diameters. In the proofs of Lemma 4.1, Lemma 4.2, and Theorem 4.4 we sometimes require an arc to end at a certain supremum. We also do so in the present proof. But, among all such arcs ending at the supremum, we choose one of small diameter. We now explain why this additional condition may replace HLC.

As in the proof of Lemma 4.1 we construct an infinite sequence $P_1, P_2, \ldots$ of simple arcs whose union is either an infinite fan or an infinite comb, that is a simple semi-arc (which we call the spine) together with a collection of pairwise disjoint simple arcs starting at the spine such that the terminating points converge towards t. We now claim that the diameter of P_n tends to zero as n tends to infinity. For suppose, *reductio ad absurdum*, that infinitely many P_n have diameter at least 7, say. Consider now a connected neighbourhood U of t contained in a compact neighbourhood of diameter at most 3. For each of the arcs P_i of diameter at least 7 intersecting U we follow P_i backwards (from its end close to t) and stop the first time the arc hits a point of distance 2 from t. Then these hitting points have a subsequence converging to t', say, and now an arcwise connected neighbourhood around t' of diameter at most 1 will give a contradiction to the minimality property of the arcs P_i since some arcs P_i of diameter at least 7 can be replaced with some of diameter less than 6.

In Lemmas 4.1, 4.2 we also encounter an augmenting semi-arc having t as an accumulation point. If it does not converge towards t, then we use the same argument as in the previous paragraph to obtain a contradiction. For, the local arcwise connectedness together with the suprema show that the diameters of the new arcs added to the current configuration tend towards zero.

This completes the proof of Theorem 5.1. ∎

Recall that a topological space is finitely separated if every point has arbitrarily small neighbourhoods with finite boundary, or equivalently, if every closed subspace can be separated from any (fixed) point it does not contain by a finite set.

In the proof of Theorem 5.1 a minimality condition on the new arcs added combined with local compactness is used to ensure that the diameters of the new arcs added to a current configuration tend to zero. For finitely separated spaces this property of the diameters is automatically satisfied as the finite set of boundary points can only be part of finitely many of the added arcs. So for any neighbourhood around t, there are only finitely many of the new arcs which are not inside this neighbourhood. So none of the arguments based on HLC or local compactness are needed for finitely separated spaces. Therefore we get the following.

Theorem 5.2. *Let s,t be distinct points in a finitely separated metric space M. Assume that, for every set S consisting of at most k elements, all distinct from s,t, $M \setminus S$ has a simple arc from s to t. Then M has $k+1$ simple arcs from s to t which have only s,t in common, pair by pair.*

In all the results preceding Theorem 5.2 we assume explicitly or implicitly that the space under consideration is locally arcwise connected. Arcs in small neighbourhoods are essential in the proofs. So, the finitely separated metric spaces are perhaps interesting in that they are the only metric spaces mentioned so far which satisfy Menger's Theorem although they need not be arcwise connected. Of course they must contain some arcs (since otherwise Menger's Theorem is void) but we do not use any other arcs in the proof for finitely separated spaces than those which are guaranteed by the assumption of Menger's Theorem.

Theorem 2.5 implies that every graph-like metric continuum and hence also each subspace of a graph-like metric continuum is finitely separated. So we obtain the following.

Theorem 5.3. *Let s,t be distinct points in a space M' which is a subspace of a graph-like metric continuum M. Assume that, for every set S consisting of at most k elements, all distinct from s,t, $M \backslash S$ has a simple arc from s to t. Then M has $k+1$ simple arcs from s to t which have only s,t in common, pair by pair.*

If M' is the Freudenthal compactification of a locally finite graph G, we may take M to consist of G and just one end. This special case of Theorem 5.3 was considered by Halin [18].

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